**STAT0041: Stochastic Calculus**

# Homework 1 - Preliminaries

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1. 设$\xi_n$是定义在概率空间$(\Omega, \mathscr{F}, \mathrm{P})$上的单调随机变量序列，且$\xi_n$依概率收敛于$\xi$. 证明：$\xi_n$几乎处处收敛于$\xi$.

2. (Jensen不等式) 设$\xi$是概率空间$(\Omega, \mathscr{F}, \mathrm{P})$上的可积随机变量，$\mathscr{G}$是$\mathscr{F}$的子事件域，$\phi$是凸函数且$\phi(\xi)$可积。证明：

$$\phi(\mathbb{E}[\xi|\mathscr{G}]) \leq \mathbb{E}[\phi(\xi)|\mathscr{G}].$$

3. 设$p \geq 1$, $\{\xi_n, n = 1, 2, \ldots\}$, $\xi \subset L^p(\mathscr{F})$, 且$\xi_n \xrightarrow{L^p} \xi$. $\mathscr{G} \subset \mathscr{F}$是一个子-$\sigma$代数。证明：

$$\mathbb{E}[\xi_n|\mathscr{G}] \xrightarrow{L^p} \mathbb{E}[\xi|\mathscr{G}].$$

Hint: 运用上一题结论。

4. 设$\xi, \eta \in L^2(\Omega, \mathscr{F}, \mathrm{P})$, 若$\xi, \eta$同分布，且$\mathbb{E}[\xi|\eta] = \eta$. 证明：$\xi = \eta$

5. (Doob不等式) 设$\xi$是filtered概率空间$(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$上的可积随机变量，证明：对任意$c > 0$,

$$P(\sup_n |\mathbb{E}[\xi|\mathscr{F}_n]| > c) \leqslant \frac{1}{c}\mathbb{E}[|\xi|].$$

Hint: consider event

$$A_n := \{\omega : \xi_1 \leqslant c, \cdots, \xi_{n-1} \leqslant c, \xi_n > c\},$$

$$A = \{\sup_n |\mathbb{E}[\xi|\mathscr{F}_n]| > c\}.$$

那么有$A = \sum_{n=1}^{\infty} A_n$.